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Bandwidth Limitations in Pulse Code Modulation

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The calculations of Huang and Reiss for the power spectrum of a pulse code modulation message is here extended, corrected, and generalized to include a formalism for an arbitrary Markovian process. The formalism contains arbitrary pulse shape and transmission times. For a first-order Markovian message the importance of the quasidiscrete frequencies is emphasized and it is concluded that these frequencies are more significant in defining the necessary bandwidth for transmission than any "optimized" choice of transmission times.

KEY WORDS: Information theory; pulse code modulation; transmission time; bandwidth; statistical mechanics; power spectra; Markovian processes.

1. INTRODUCTION

The relationship between thermodynamics and information theory is well established.(1) Recently Reiss(2) and Reiss and Huang(3) have investigated the application of statistical thermodynamics to information theory. In the latter paper a general formalism was developed for compact codes, and Huang and Reiss(4) applied this formalism to some specific examples of binary codes with and without memory.

The attempt in this last paper was to optimize (i.e., minimize) the transmission time–bandwidth product. Huang and Reiss came to the con-

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clusion that their prescription of optimizing the transmission time per bit did not in all cases lead to a reduction of transmission time–bandwidth product compared to that which is obtained with a fixed transmission time. Since this conclusion appears somewhat surprising, the present investigation was initiated to review the concepts which underlay that result and to provide a better understanding of it. In fact, we find that the conclusion was the result of a mathematical error and that a correct analysis indeed leads to the expected result.

In this paper we shall first briefly review the Huang–Reiss concepts, then extend and generalize them to a pulse-code modulation message based on an n-letter alphabet with arbitrary Markovian memory. This general formalism will then be reduced to the specific instance of a binary code and applied to the examples investigated by Huang and Reiss. Since our conclusions differ from those of Huang and Reiss, we shall briefly trace their calculations in order to identify the origin of their error.

2. STATISTICAL THERMODYNAMIC FORMALISM

Reiss and Huang[3] consider the transmission of a binary PCM message (a sequence of pulses representing zeros and ones). They show that maximum information can be transmitted in a given transmission time if the length of the pulse transmission time is matched to the statistics of the message. Based on statistical thermodynamic arguments and an isomorphism between thermodynamics and information theory (which leads to information analogs of temperature, pressure, entropy, free energy,...) they define an information theory partition function. From this they deduce the optimum transmission time τ_i for a given bit (0 or 1) to be

$$\tau_i = \tau \ln p_i \tag{1}$$

where p_i is the probability of the bit b_i in the "language" constituting the message.

For correlated messages, ones in which the probability that a given bit will be transmitted is influenced by the bits previously transmitted, a more general formulation must be used. In this case the optimum transmission time for a particular bit $\tau(i|S)$ depends on the conditional probability $p(i|S)$ that that bit will be used at that given position in the message

$$\tau(i|S) = -\tau \ln p(i|S) \tag{2}$$

where $p(i|S)$ is the conditional probability of the bit b_i appearing as the next bit in a message whose "state" is S. For a finite Markovian system the state S is defined by a finite sequence of preceding bits.

In the third paper of the series, Huang and Reiss[4] introduce the Wiener–Khinchine theorem,[5] which relates the autocorrelation function of the message $\phi(t)$ to the power spectrum $G(\omega)$,

$$G(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}\phi(t)\, dt \tag{3a}$$

in which ω is the angular frequency and t is the displacement time in the autocorrelation function. We of course also have the inverse relationship

$$\phi(t) = (1/2\pi)\int_{-\infty}^{\infty} G(\omega)e^{-i\omega t}\, d\omega \tag{3b}$$

These equations are familiar in other branches of physics as well. In atomic and molecular physics the correlation function is that of the electric dipole operator and $G(\omega)$ is the spectral line shape of the absorbed or emitted radiation, or $\phi(t)$ is the autocorrelation function of the dynamics of molecular motion (position and time) and $G(\omega)$ is directly related to the cross section for momentum and energy transfer in neutron scattering by the molecular system.[6]

3. GENERAL POWER DENSITY SPECTRUM

The autocorrelation function of a message is defined by the integral

$$\begin{aligned} \phi(t) &= \lim_{T\to\infty} \frac{1}{T-t}\int_0^{T-t} S(\tau)S(t+\tau)\, d\tau, \qquad t > 0 \\ &= \lim_{T\to\infty} \frac{1}{T+t}\int_{-t}^{T} S(\tau)S(t+\tau)\, d\tau, \qquad t < 0 \end{aligned} \tag{4}$$

where $S(\tau)$ is the pulse intensity at time τ. The limit implies that it is possible to transmit an infinitely long message from which the statistics can be determined. If only a finite number of messages are possible, then the limit actually terminates at a time T required to send all possible messages. However, since a sequence of m messages transmitted in succession is itself usually a possible message, the "set of all possible messages" is usually infinite and requires an infinite transmission time. An alternative to (4) is the ensemble average

$$\phi(t) = \left\langle \frac{1}{T-t}\int_0^{T-t} S(\tau)S(\tau+t)\, d\tau \right\rangle \tag{5}$$

where T is now the length of a single message and the average is an average over all possible messages. In a strict operational sense (5) is just as difficult to carry out as (4), but it usually easier to conceptualize. Since the possible number of long messages is very much greater than the number of short messages, the average is dominated by the statistics of messages for which $T \to \infty$ and hence we achieve (4) from (5) in the mean.

In a strict sense $\phi(t)$ is usually normalized by dividing expression (4) by its value at $t = 0$, i.e., by

$$\phi(0) = \langle (1/T) \int_0^T S^2(\tau)\, d\tau \rangle \tag{6}$$

but for our present purposes we make no use of this normalization and we shall therefore omit it. Of more importance, $S(t)$ is properly defined as the variation of the transmitted signal from the mean transmitted intensity, so that (in the absence of long-range correlations) one has

$$\lim_{t=\infty} \phi(t) = 0 \tag{7}$$

This suppresses the appearance of a delta function at $\omega = 0$ in the power spectrum $G(\omega)$. We shall therefore assume that $S(\tau)$ is so defined and hence

$$\langle (1/T) \int_0^T S(\tau)\, d\tau \rangle = 0 \tag{8}$$

We point out that (8) does not imply (7), and that the limit in (7) may not exist if there are long-range correlations in $S(\tau)$. The simplest and most common type of long-range correlation in a message is the appearance of periodicities in $S(\tau)$. These arise primarily from the artificial, but common, approximation of square pulses with equal (or rationally related) transmission times. In such cases the power spectrum consists of a discrete set of frequencies (delta functions) as well as the continuum spectrum. We shall return to this point later.

We now describe the message in terms of an alphabet of letters b_j. In a binary code there are only two letters $b_1 = 0$, $b_2 = 1$; in English there are at least 27 characters,[2] $b_1 = a$, $b_2 = b$, $b_3 = c$, etc. A message is a sequence of letters $b_{j_1} b_{j_2} b_{j_3} \cdots$ and can then be characterized by the ordered set $\{j_1, j_2, j_3, \ldots, j_k, \ldots\}$. Since each message corresponds to a different sequence of letters, the mth message is denoted by the set $\{j_{1m}, j_{2m}, j_{3m}, \ldots, j_{N_m,m}\}$ where N_m is the number of letters in the mth message. (The "message" in the sense used here is the total set of symbols transmitted and includes all identifiers, punctuation, and redundancy checks which may be transmitted.)

We shall denote the signal or pulse associated with the letter b_j as $s(t; j)$, measured from some arbitrary zero of time. Usually the signal will be such that $s(t; j) = 0$ for $t < 0$ but it is not essential that the time origin

[2] We include a blank space and all punctuation symbols which may be transmitted in a message as "characters."

associated with a signal pulse satisfy this condition. Since the message is transmitted as a sequence of pulses, each pulse must be allowed a certain transmission time and hence the transmission of the signal for the kth letter requires a delay of σ_{j_k} before the next pulse can be transmitted.[3] Thus the transmitted signal corresponding to the sequence $\{j_{k,m}\}$ is

$$S(t) = \sum_k s(t - t_{k,m}; j_{k,m}) \tag{9a}$$

where

$$t_{1,m} = 0, \qquad t_{k+1,m} = t_{k,m} + \sigma_{j_{k,m}} \tag{9b}$$

Substituting (9a) into (5) gives

$$\phi(t) = \left\langle \frac{1}{T-t} \int_0^{T-t} \sum_{k,k'} s(\tau - t_k; j_k) s(\tau + t - t_{k'}; j_{k'})\, d\tau \right\rangle \tag{10}$$

The power spectrum is then

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega t} \phi(t)\, dt \\ &= \left\langle \int_{-\infty}^{\infty} dt \lim_{T\to\infty} \frac{1}{T} \int_0^T d\tau \sum_{k,k'} s(\tau - t_k; j_k) s(\tau + t - t_{k'}; j_{k'}) e^{-i\omega t} \right\rangle \end{aligned} \tag{11}$$

If we now put $\tau - t_k = u$ and $\tau + t - t_{k'} = v$, this becomes

$$\begin{aligned} G(\omega) &= \lim \left\langle \frac{1}{T} \sum_{k,k'} \int s(u; j_k) \exp(i\omega u)\, du \right. \\ &\qquad \times \left. \int s(v; j_{k'}) \exp(-i\omega v)\, dv \exp[-i\omega(t_{k'} - t_k)] \right\rangle \\ &= \lim \left\langle \frac{1}{T} \sum_{k,k'} g_{j_k}^*(\omega) [\exp(i\omega t_k)] g_{j_{k'}}(\omega) \exp(i\omega t_{k'}) \right\rangle \end{aligned} \tag{12}$$

$g_j(\omega)$ is the Fourier transform of the pulse transmitted for the jth symbol of the alphabet. Since the average is taken over all messages, (12) reduces to an average over the language:

$$G(\omega) = (1/\langle\sigma_i\rangle)[\langle|g_i(\omega)|^2\rangle + 2\,\mathrm{Re}\langle g_i^*(\omega) \sum_{s=1}^{\infty} T_{sij} g_j(\omega)\rangle] \tag{13}$$

[3] It is also possible to consider that the time delay between the kth and the $(k+1)$th pulses depends on the two characters b_k and b_{k+1}, but we shall ignore this minor complication, and assume that the delay between two pulses depends only on the preceding character.

where

$$T_{sij} = e^{-i\omega t_s} \tag{13a}$$

and

$$t_s = t_{k'} - t_k = \sum_{r=0}^{s} \sigma_r \tag{13b}$$

The time t_s is the total displacement in time between the transmission of the symbol b_i and the sth following symbol.

Equation (13) is a much more convenient and more general expression than could be developed following the Huang–Reiss procedure. In their work the correlation function itself is explicitly evaluated and then the Fourier transform yields the power spectrum. As a practical matter the correlation function can be easily evaluated only when the transmission times of the individual pulses are rationally related. The complexity of this process, however, is apparent even in the simple case of a binary code with two pulse lengths in the ratio 1:2. For an arbitrary code the total time span which would need to be considered in constructing the correlation function would have to be the least common multiple of the transmission times of the set of message pulses. Furthermore, if the pulse shape were anything more complex than square pulses of arbitrary amplitude, the algebraic structure of the correlation function would be replaced by expressions involving convolution integrals of the pulse shapes. These complexities are essentially bypassed in Eq. (13), which expresses the power spectrum directly in terms of the Fourier transforms of the individual pulses $g_j(\omega)$ (containing the process of implementation of the message into a transmitted signal) and the time displacement matrix T_{sij} (containing the statistics of the jth symbol of the alphabet appearing s positions after the ith symbol).

Equation (13) is the general expression underlying the calculation of the power spectrum. In order to actually utilize this expression we must be able to evaluate, at least in a statistical sense, the time displacement function t_s. The extent to which this can be done depends on the extent to which one knows the statistical structure of the language.[4] For the purposes of calculation of power spectra the structure of the language is defined by the conditional probability that a letter b_k appears in the language given the preceding sequence $\{j_1, j_2, \ldots, j_l\}$. Uncorrelated messages imply that the probability of appearance of a given letter is independent of the previous sequence. If the minimum length of sequence required to completely define the statistics of

[4] The statistics of messages of course depends on the universe of discourse. The frequency of occurrence of particular symbols is dependent on the subject matter being transmitted. In the information theory sense, then, physicists and lawyers speak different languages.

the message is k, then the language constitutes a kth-order Markovian process.

The simplest situation is, of course, the uncorrelated, or zeroth-order Markovian, process. The time displacement factor in (13) is independent of i and can then be written

$$T_{sj} = \sum p_k T_{s-1,k} e^{-i\omega\sigma_j} = \langle T_{s-1}\rangle e^{-i\omega\sigma_j} \tag{14}$$

where p_k is the probability of appearance of the kth letter and σ_k is its transmission time. From this one obtains

$$\langle T_s\rangle = \langle T_{s-1}\rangle\langle e^{-i\omega\sigma_j}\rangle \tag{14a}$$

$$T_{sj} = e^{-i\omega\sigma_j}\langle e^{-i\omega\sigma}\rangle^{s-1} \tag{14b}$$

and

$$G(\omega) = \frac{1}{\langle\sigma\rangle}\left[\langle|g(\omega)|^2\rangle + 2\,\mathrm{Re}\,\frac{\langle g^*(\omega)e^{-i\omega\sigma}\rangle\langle g(\omega)\rangle}{1 - \langle e^{-i\omega\sigma}\rangle}\right] \tag{15}$$

We may also note that (15) may be simplified notationally even further by setting

$$\bar{g}(\omega) = g^*(\omega)e^{-i\omega\sigma} = \int_{-\infty}^{\infty} s(\sigma - t)e^{-i\omega t}\,dt \tag{16}$$

so that $\bar{g}(\omega)$ is the Fourier transform of the pulse measured backward from its end, in distinction to $g(\omega)$, which is measured forward from the start. If the pulse is symmetric, we then have $\bar{g}(\omega) = g(\omega)$.

For a correlated message (15) may be generalized

$$G(\omega) = (1/\langle\sigma\rangle)\langle|g(\omega)|^2\rangle + 2\,\mathrm{Re}\langle g^* M(1 - M)^{-1} g\rangle \tag{17}$$

where g is a vector and M is a matrix of displacement factors. For the uncorrelated process g is just the vector of the Fourier transforms of the pulses which make up the alphabet and M is the matrix

$$M_{jk} = e^{-i\omega\sigma_j} p_k \tag{18}$$

In this case (17) reduces to (15). For a nth-order Markov process the transmitted pulse may depend not only on the symbol being transmitted but on the state of the system, i.e., the n preceding symbols as well as the symbol presently being transmitted. The "state vector" therefore consists of an ordered sequence of $n + 1$ elements and the probability of the system emitting a symbol characterized by the last index is conditional upon the specification of the first n indices. In this case the transmitted pulse is identified by $n + 1$ indices and the vector g contains r^{n+1} elements, where r is the number of symbols in the alphabet. Thus g is actually an $(n + 1)$th-rank tensor which is mapped into a vector. Equation (18) is then still valid if p_k is replaced by the state transition probability which takes the system from state j to state k. This transition probability vanishes, of course, if the last n elements of the initial state do not correspond to the first n elements of the final state. Thus, if the state vector S is the sequence $\{abc \cdots f\}$, the transition probability $P(S \to S')$ vanishes unless S' is the sequence $\{bcd \cdots fg\}$, in which case the transition probability is the conditional probability $P(S \to S') = p(g|bcd \cdots f)$.

The most convenient mapping of the state vector into a scalar (and hence the g tensor into a vector) is to consider the sequence $abc \cdots j$ as the representation of an integer in a base-r number system. For an nth-order Markovian process one then can write $\alpha = ar^n + br^{n-1} + \cdots + j$ and

$$P(S \to S') = P(\alpha \to \beta) \equiv P_{\alpha\beta}$$
$$= \delta(r \cdot \alpha \bmod r^n + \beta \bmod r - k) p(\beta \bmod r | \alpha \bmod r^n) \quad (19)$$

where $\delta(x)$ is the Kronecker delta: $\delta(0) = 1$; $\delta(x) = 0$, $x \neq 0$.

We shall make no specific use of this completely general formalism except to apply it to the case of a binary first-order Markovian process in Section 5.

4. APPLICATION TO BINARY CODES

The binary code consists of two symbols, which we may take as 0 and 1. It is simplest to consider a coding in which 0 is represented by the absence of a pulse and a 1 by its presence. Huang and Reiss[4] considered only rectangular pulses, but we shall not specify the shape yet, leaving it arbitrary with a transform $g_1(\omega)$. If the probability of a 0 is p_0 and that of a 1 is p_1 (with $p_0 + p_1 = 1$), we have

$$\langle \sigma \rangle = p_0\sigma_0 + p_1\sigma_1 \quad (20a)$$

$$\langle |g(\omega)|^2 \rangle = p_1 |g_1(\omega)|^2 \quad (20b)$$

and

$$\langle e^{-i\omega\sigma} \rangle = p_0 e^{-i\omega\sigma_0} + p_1 e^{-i\omega\sigma_1} \quad (20c)$$

where σ_0 and σ_1 are the transmission times for the respective signals. Then, from (15)

$$G(\omega) = \frac{p_0 p_1 |g_1(\omega)|^2}{p_0\sigma_0 + p_1\sigma_1} \times \frac{1 - \cos\omega\sigma_0}{p_0(1 - \cos\omega\sigma_0) + p_1(1 - \cos\omega\sigma_1) - p_0 p_1[1 - \cos\omega(\sigma_0 - \sigma_1)]} \quad (21)$$

If $\sigma_0 = \sigma_1$ (fixed transmission rate), this reduces to

$$G(\omega) = (p_1 p_0/\sigma)|g_1(\omega)|^2 \tag{22}$$

which is a useful generalization of Huang and Reiss' result and reduces to it when $|g_1(\omega)|^2$ is evaluated for a square pulse. However, we do not recover, in this formulation, the delta function which Huang and Reiss obtain. We can achieve this if the various limiting processes implicit in reaching (15) are carried out more carefully. On the other hand, as indicated above, we should define our symbol pulses in such a way that the time average over all messages is zero. Thus $g_1(\omega)$ should be replaced by

$$g_1(\omega) - p_1 \frac{1 - e^{-i\omega\sigma_1}}{i\omega\langle\sigma\rangle} g_1(0)$$

and $g_0(\omega)$, instead of representing the absence of any signal, should be considered to be

$$-p_1 \frac{1 - e^{-i\omega\sigma_1}}{i\omega\langle\sigma\rangle} g_1(0)$$

The expedient of simply ignoring, in the power spectrum, the δ-function at zero frequency accomplishes the same end result.

Huang and Reiss also consider an explicit nonconstant transmission time case with $\sigma_0 = 2\sigma_1$. In this case we obtain

$$G(\omega) = \frac{4p_0}{\sigma_1} \frac{1 - p_0}{1 + p_0} \frac{|g_1(\omega)|^2(1 + \cos \omega\sigma_1)}{1 + 2p_0 \cos \omega\sigma_1 + p_0^2} \tag{23}$$

which does not agree with the Huang–Reiss result. In the appendix we review the Huang–Reiss analysis and show, in fact, that errors were made in those calculations.

For a rectangular pulse of amplitude unity and width σ_1 we have the well-known transform

$$g(\omega) = \frac{1 - e^{-i\omega\sigma_1}}{i\omega} \tag{24a}$$

and

$$|g(\omega)|^2 = \frac{2}{\omega^2}(1 - \cos \omega\sigma_1) = [(2/\omega) \sin (\omega\sigma_1/2)]^2 \tag{24b}$$

If the bandwidth of the power spectrum is defined as the width at half maximum, then (22) has a transmission time–bandwidth product given by $\sigma\omega_{1/2} = 2.7831$. For nonequal transmission times we have from (23)

$$G(\omega) = 4\sigma_1 p_0 \frac{1 - p_0}{1 + p_0} \left(\frac{\sin \omega\sigma_1}{\omega\sigma_1}\right)^2 [1 + 2p_0 \cos \omega\sigma_1 + p_0^2]^{-1} \tag{25}$$

where the "optimized" transmission rate (with $\sigma_0 = 2\sigma_1$) corresponds to $p_0 = 0.381966$. The mean bit transmission rate is $(1 + p_0)\sigma_1$ and we find

$$(1 + p_0)\sigma_1\omega_{1/2} = 2.7008$$

which is a reduction of approximately 3% compared to the case of constant transmission rate.

If, however, we return to (21) and, with p_0 fixed, vary the ratio σ_0/σ_1, we find that $\langle\sigma\rangle\omega_{1/2}$ has a minimum value 2.6538 (a reduction of 5% below the uniform transmission rate value). This minimum occurs for $\sigma_0/\sigma_1 = 1.509$ and hence corresponds to a transmission time ratio which is not as large as the ratio corresponding to the thermodynamic optimum ratio $\sigma_0/\sigma_1 = 2$.

Figure 1 shows, for $p_0 = 0.381966$, the transmission time–bandwidth product for a binary coded message. Bandwidth is defined as the value $\omega_{1/2}$, $[G(\omega_{1/2}) = \frac{1}{2}G(0)]$, calculated from (21) with $|g_1(\omega)|^2$ corresponding to a square pulse. The small circles indicate the time–bandwidth product for $\sigma_0 = \sigma_1$ $(\sigma_1/\langle\sigma\rangle = 1)$ and for $\sigma_0 = 2\sigma_1$ $(\sigma_1/\langle\sigma\rangle = 0.7236)$. The dashed line is a similar curve for a Gaussian pulse shape, i.e., $|g_1(\omega)|^2 = \exp -\frac{1}{12}(\omega\sigma)^2$.

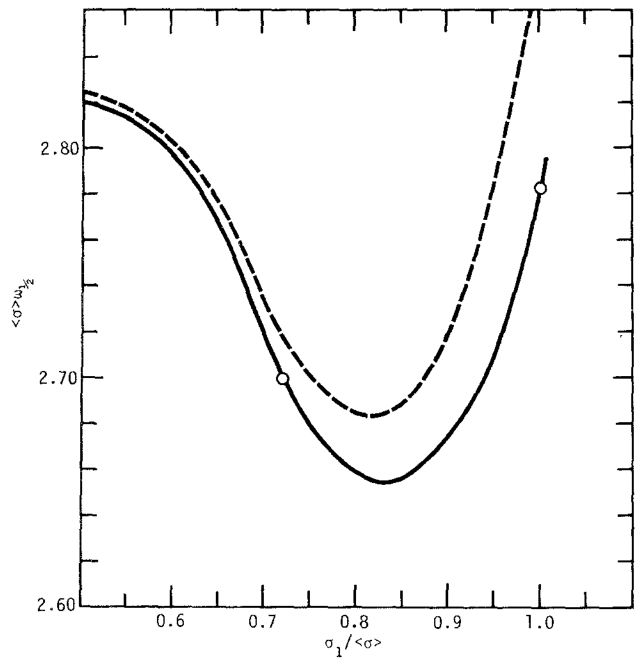

Fig. 1. Time–bandwith product for an uncorrelated message, $p_1 = (\sqrt{5} - 1)/2 = 0.618034$. The width is defined as the frequency at half height of the power spectrum. Solid line: square pulse; dashed line: Gaussian pulse, $|g_1(\omega)|^2 = \exp[-(\omega\sigma)^2/12]$. The two open circles identify the cases $\sigma_0 = \sigma_1$ and $\sigma_0 = 2\sigma_1$ $(\sigma_1/\langle\sigma\rangle = 0.7236)$.

The width of the Gaussian has been taken to be such that it has the same second moment as a square pulse and hence the power spectra of the two pulses have the same second derivative at $\omega = 0$.

5. BINARY CODE WITH MEMORY

When the probability of the appearance of a letter of the alphabet depends conditionally on the sequence of preceding letters one has a Markovian process. We shall now extend our analysis to the case of a first-order process in which the probability of occurrence of a 0 or a 1 depends on the value of the preceding bit. The extension to higher order Markov process and to larger alphabets will be obvious.

In the binary first-order process there are four conditional probabilities: p_{00}, p_{01}, p_{10}, and p_{11}, where p_{ij} is the conditional probability that bit j will occur when the preceding bit is known to be i. In more common conditional probability notation

$$p_{ij} = P(j|i) \tag{26}$$

In addition we also define the unconditional probabilities P_0 and P_1, the probabilities that a 0 or a 1 occurs in the message ensemble. These probabilities are not independent; we have the following relationships:

$$\sum_i P_i p_{ij} = P_j \tag{27a}$$

$$\sum_j p_{ij} = 1 \tag{27b}$$

$$\sum_i P_i = 1 \tag{27c}$$

This set of equations, which is generally valid for any first-order Markovian process, defines the state probabilities P_i in terms of the off-diagonal transition probabilities p_{ij} $(i \neq j)$. These off-diagonal elements may be taken as the defining independent parameters of the Markovian process. For the binary case in particular we have

$$P_0 = \frac{p_{10}}{p_{10} + p_{01}}, \qquad P_1 = \frac{p_{01}}{p_{10} + p_{01}} \tag{27d}$$

and

$$P_{00} = \frac{p_{00}p_{10}}{p_{01} + p_{10}}, \qquad P_{01} = P_{10} = \frac{p_{01}p_{10}}{p_{01} + p_{10}}, \qquad P_{11} = \frac{p_{01}p_{11}}{p_{01} + p_{10}} \tag{27e}$$

To construct the time displacement matrix M we identify the four state indices (00, 01, 10, 11) that define the conditional probabilities with the matrix indices 0, 1, 2, 3. In order to distinguish between the two forms of enumeration we shall use Greek letters to indicate the matrix index ($\alpha = 0, 1, 2, 3$) and Latin letters for the state index ($i = 0, 1$). With the transition probabilities defined by (19), M then has the form

$$M = \begin{pmatrix} p_0 e^{-i\omega\sigma_0} & p_1 e^{-i\omega\sigma_0} & 0 & 0 \\ 0 & 0 & p_2 e^{-i\omega\sigma_1} & p_3 e^{-i\omega\sigma_1} \\ p_0 e^{-i\omega\sigma_2} & p_1 e^{-i\omega\sigma_2} & 0 & 0 \\ 0 & 0 & p_2 e^{-i\omega\sigma_3} & p_3 e^{-i\omega\sigma_3} \end{pmatrix} \tag{28}$$

and we find

$$\langle g^* M(1 - M)^{-1} g\rangle = (1/K)\langle g_\alpha{}^* e^{-i\omega\sigma_\alpha} Q p_\beta g_\beta\rangle_\alpha \tag{29a}$$

where, with $\xi_\alpha = e^{-i\omega\sigma_\alpha}$, we have

$$Q = \begin{pmatrix} 1 - p_3\xi_3 & 1 - p_3\xi_3 & p_1\xi_1 & p_1\xi_1 \\ p_2\xi_2 & p_2\xi_2 & 1 - p_0\xi_0 & 1 - p_0\xi_0 \\ 1 - p_3\xi_3 & 1 - p_3\xi_3 & p_1\xi_1 & p_1\xi_1 \\ p_2\xi_2 & p_2\xi_2 & 1 - p_0\xi_0 & 1 - p_0\xi_0 \end{pmatrix}$$

$$K = (1 - p_0\xi_0)(1 - p_2\xi_2) - p_1 p_2 \xi_1 \xi_2 \tag{29b}$$

The second term in (13), which we can write symbolically as 2 Re$\langle g^* e^{-i\omega t_s} g\rangle$ can then be expressed as

$$\langle g^* e^{-i\omega t_s} g\rangle = (1/K) \sum_{\alpha\beta} P_\alpha g_\alpha{}^* \xi_\alpha Q_{\alpha\beta} p_\beta g_\beta \tag{30}$$

When a 1 is transmitted as a pulse and a 0 as blank, then $g_{00}(\omega) = g_0(\omega) = 0$ and $g_{10}(\omega) = g_2(\omega) = 0$ and the expression becomes

$$\langle g^* e^{i\omega t_s} g\rangle = \frac{1}{K}(P_{01} g_1{}^* \xi_1 + P_{11} g_3{}^* \xi_3)[p_2\xi_2 p_1 g_1 + (1 - p_0\xi_0) p_3 g_3]$$

$$= \frac{p_1(p_2\xi_1 g_1{}^* + p_3\xi_3 g_3{}^*)[p_2\xi_2 p_1 g_1 + p_3(1 - p_0\xi_0) g_3]}{(p_1 + p_2)[(1 - p_0\xi_0)(1 - p_3\xi_3) - p_1 p_2 \xi_1 \xi_2]} \tag{31}$$

and

$$\langle\sigma\rangle G(\omega) = \frac{p_1}{p_1 + p_2} \operatorname{Re} \frac{A_{11} g_1{}^* g_1 + A_{13} g_1{}^* g_3 + A_{31} g_3{}^* g_1 + A_{33} g_3{}^* g_3}{(1 - p_0\xi_0)(1 - p_3\xi_3) - p_1 p_2 \xi_1 \xi_2} \tag{31a}$$

where

$$A_{11} = p_2[(1 - p_0\xi_0)(1 - p_3\xi_3) + p_1 p_2 \xi_1 \xi_2]$$

$$A_{13} = 2p_2 p_3 (1 - p_0\xi_0)\xi_1, \qquad A_{31} = 2p_1 p_2 p_3 \xi_2 \xi_3$$

$$A_{33} = p_3[(1 - p_0\xi_0)(1 + p_3\xi_3) - p_1 p_2 \xi_1 \xi_2]$$

If there is no correlation between successive signals, we have $p_0 = p_2$, $p_1 = p_3$ and $\xi_0 = \xi_2$, $\xi_1 = \xi_3$, and (31) reduces to $p_1{}^2|g|^2\xi_1(1 - p_0\xi_0 - p_1\xi_1)^{-1}$, which leads again to Eq. (21).

For square pulses we introduce (24a) and (24b) into (31) and make use of the identity $|1 - \xi|^2 = 2\,\mathrm{Re}(1 - \xi)$ for $|\xi| = 1$; this reduces (13) to the form

$$\omega^2\langle\sigma\rangle G(\omega) = \frac{2p_1p_2}{p_1 + p_2}\,\mathrm{Re}\,\frac{(1 - p_0\xi_0 - p_1\xi_2)(1 - p_2\xi_1 - p_3\xi_3)}{(1 - p_0\xi_0)(1 - p_3\xi_3) - p_1p_2\xi_1\xi_2} \qquad (32)$$

In order to compare this with the Huang–Reiss condition we now specify $p_{01} = p_{10}$ $(p_1 = p_2)$ and $p_{00} = p_{11}$ $(p_0 = p_3)$. Furthermore, we set $\sigma_{01} = \sigma_{10}$ $(\sigma_1 = \sigma_2)$ and $\sigma_{00} = \sigma_{11}$ $(\sigma_0 = \sigma_3)$, which implies also $\xi_1 = \xi_2$ and $\xi_0 = \xi_3$. For this case the denominator in (32) factors into two terms, one of which cancels one of the terms in the numerator, and we then obtain

$$\begin{aligned}\langle\sigma\rangle G(\omega)\omega^2 &= p_1\,\mathrm{Re}\,\frac{1 - p_1\xi_1 - p_3\xi_3}{1 + p_1\xi_1 - p_3\xi_3}\\ &= \frac{p_1p_3(1 - \cos\omega\sigma_3)}{p_1(1 + \cos\omega\sigma_1) + p_3(1 - \cos\omega\sigma_3) - p_1p_3[1 + \cos\omega(\sigma_3 - \sigma_1)]}\end{aligned} \qquad (33)$$

With a constant transmission rate, $\sigma_1 = \sigma_3 = \sigma$, this reduces further to

$$\sigma G(\omega)\omega^2 = \frac{2p_1p_3(1 - \cos\omega\sigma)}{1 + 2(p_1 - p_3)\,\mathrm{cps}\,\omega\sigma + (p_1 - p_3)^2} \qquad (34)$$

which is one of the cases considered by Huang and Reiss. They also consider the situation in which σ_3 is twice σ_1; this leads to $\langle\sigma\rangle = (1 + p_3)\sigma_1$ and

$$G(\omega) = \frac{2p_1p_3(1 - \cos\omega\sigma_1)}{(1 + p_3)\sigma_1\omega^2[1 - 2p_3(\cos\omega\sigma_1) + p_3{}^2]} \qquad (\sigma_3 = 2\sigma_1) \qquad (35)$$

The Reiss prescription for the "optimum" transmission rate is $\sigma_{ij} = -k \ln p_{ij}$ and hence for $\sigma_3 = 2\sigma_1$ we have $p_1{}^2 = p_3$ or $p_1 = \frac{1}{2}(\sqrt{5} - 1) = 0.61803$, $p_3 = 0.38197$. The normalized power spectrum $G(\omega)/G(0)$ is shown in Fig. 2 for this value of p_1. The dimensionless abscissa in this figure is the frequency times the mean transmission time. The power spectrum is calculated for several values of the ratio σ_1/σ_3 between $\frac{1}{2}$ and 1. Because of the correlation in the message the power spectrum does not have its maximum at $\omega = 0$. This peaking of the power spectrum is very marked at $\sigma_1/\sigma_3 = 0.6$ but has disappeared in Eq. (35), which corresponds to $\sigma_1/\sigma_3 = 0.5$. There are, however, "hidden" delta functions in Eq. (33) whenever $1 + \cos\omega\sigma_1$ and $1 - \cos\omega\sigma_3$ vanish simultaneously. This occurs when $\omega\sigma_1 = (2k - 1)\pi$ and $\omega\sigma_3 = 2m\pi$, where k and m are integers. In that case $1 + \cos[\omega(\sigma_3 - \sigma_1)]$ will also vanish. Hence we obtain a delta function at $\omega\langle\sigma\rangle = \pi[(2k - 2m - 1)p_1 + 2m]$. The

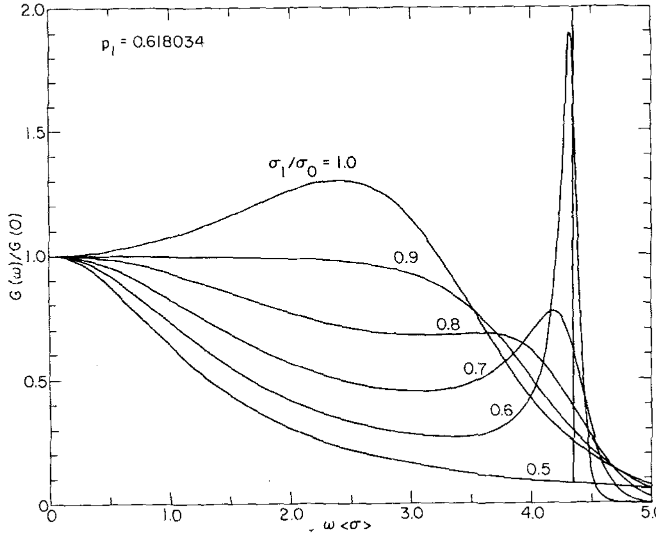

Fig. 2. Power spectrum for a first-order Markovian process, $p_{10} = p_{01} = (\sqrt{5} - 1)/2$, square pulse shape [Eq. (33)]. For $\sigma_{11} = 2\sigma_{01}$, Eq. (35) does not display the delta function, which must be determined by a limiting process $\sigma_{11} \to 2\sigma_{01}$.

case $\sigma_3 = 2\sigma_1$ corresponds to $m = 2k - 1$ and hence produces delta functions at $\omega\langle\sigma\rangle = \pi(1 + p_3)m$.

The resonance in Eq. (33) can be explicitly displayed by rewriting the expression in the form

$$\langle\sigma\rangle G(\omega) = \frac{p_1 p_3[(2/\omega)\sin(\omega\sigma_3/2)]^2}{(p_1 \sin\omega\sigma_1 - p_3 \sin\omega\sigma_3)^2 + (1 + p_1\cos\omega\sigma_1 - p_3\cos\omega\sigma_3)^2} \tag{36}$$

an expression which approaches the Lorentzian line shape. To see this more clearly we now introduce $\sigma_1/\sigma_3 = (2k - 1 + \epsilon)/2m$, so that, with $\epsilon \ll 1$, the two transmission times are close to the critical rational ratio. Then with $\omega\sigma_3 = 2m\pi + \sigma_3\delta$, Eq. (36) can be reduced to

$$G(\omega) = \frac{4p_3}{p_1\langle\sigma\rangle\omega^2}\left[1 + \frac{(p_1{}^2\epsilon\pi\sigma_3/2\langle\sigma\rangle)^2}{\langle\sigma\rangle^2(\delta - \delta_0)^2 + (p_1p_3\epsilon^2\pi^2\sigma_3{}^2/2\langle\sigma\rangle^2)^2}\right] \tag{37}$$

where $\delta_0 = -p_1\epsilon\pi/\langle\sigma\rangle$ gives the position of the peak (in the limit $\epsilon \to 0$). At the peak we have

$$G(\hat{\omega}) = \frac{4p_3}{p_1\langle\sigma\rangle\hat{\omega}^2}\left[1 + \left(\frac{p_1\langle\sigma\rangle}{\epsilon\pi p_3\sigma_3}\right)^2\right]$$

while the width of the peak is $\gamma = \epsilon^2\pi^2p_1p_3\sigma_3{}^2/\langle\sigma\rangle^3$. The total area under this peak is therefore finite as $\epsilon \to 0$, and is equal to $2\pi p_1{}^2/\langle\sigma\rangle^2\hat{\omega}^2$. The position of these peaks is $\hat{\omega} = 2m\pi/\sigma_3$ and since for $\sigma_3 = 2\sigma_1$, m may be any odd integer, the total integral over all of the peaks (using the summation $\sum m^{-2} = \pi^2/8$) becomes

$$\frac{\pi}{4}\left(\frac{p_1}{1 + p_1}\right)^2$$

to be compared with the total area of the power spectrum $G(\omega)\,d\omega = \pi$. The total discrete spectrum is then

$$G_d = \frac{\pi}{4}\,\delta(\omega) + \frac{2}{\pi}\left(\frac{1 - p_3}{1 + p_3}\right)^2 \sum_m \frac{1}{m^2}\,\delta\left(\omega - \frac{m\pi}{\sigma_1}\right) \tag{38}$$

which must therefore be added to the continuous spectrum, Eq. (35), to obtain the total power spectrum.

6. CONCLUSIONS

We have explored several explicit examples of pulse code modulation. The general formulation developed here for the calculation of the power spectrum of messages transmitted with an arbitrary alphabet and arbitrary symbol pulse shapes can be applied if the statistics of the language (in terms of the probabilities of occurrence of symbols in the message) can be defined. An attempt has been made to provide a recipe for optimization of transmission times of the symbols in terms of the statistical-mechanical analog arguments of Reiss and Huang. In this connection, we have noted that the *complete* problem of optimization must be referred to the transmission time–bandwidth product rather than to transmission time alone. Depending upon the definition of bandwidth (half-width of first peak, etc.) the prescription for optimization may vary. In particular, for the definition we have chosen in Fig. 1, optimization is not achieved by the Reiss–Huang prescription.

We have shown by explicit example that even with the time–bandwidth problem, no such generalization is possible because of the periodicities introduced into the signal correlation function if the transmission times of the pulses are rationally related. Quasiperiodicities exist if the ratios of transmission times are only approximately small rational fractions. Exact periodicities of course produce discrete frequencies in the power spectrum, whereas quasiperiodicities produce strong peaks in the spectrum corresponding to the frequencies of the periodicities. Since the lowest nonzero frequency peak will in general contain a significant fraction of the transmitted energy, the bandwidth required for transmission of the message will generally be limited by this feature rather than by any optimization process in terms of spectral half-width. It is also important to realize that the existence of the peaks in the power spectrum is determined by the statistics and transmission times of the symbols which constitute the message. The pulse shape, and the frequency content of the transmitted pulses, can only affect the amount of energy contained in these peaks, not their existence.

APPENDIX

In this appendix we compute directly the correlation functions for an uncorrelated and a correlated binary code. We follow in general the approach of Huang and Reiss,[4] but do not use exactly the same notations. Our primary purpose here is to demonstrate directly the discrete frequency spectrum which arises from the assumption of rationally related transmission times.

As with Huang and Reiss, we shall here assume square pulses; a square pulse of length t_1 shall represent the binary symbol 1 and an absence of signal of length t_0 shall represent a binary 0. Choosing an arbitrary point in time, the probability that this point lies within a pulse is

$$P = \frac{p_1 t_1}{p_1 t_1 + p_0 t_0} \tag{A.1}$$

where p_1 is the probability of occurrence of a 1 in the message and $p_0 = 1 - p_1$ is the probability of occurrence of a 0. For the interval $0 < \tau < t_1$ there is a probability $1 - \tau/t_1$ that the interval τ lies entirely within a single pulse and a probability τ/t_1 that it spans two message symbols. Since the probability is p_1 that the second symbol is 1, the correlation function is

$$\phi(\tau) = P[1 - (\tau/t_1) + p_1(\tau/t_1)] \tag{A.2}$$

For longer times we must specify the lengths t_1 and t_0; we choose the Huang and Reiss case, $t_0 = 2t_1 = 2\sigma$. We also denote the correlation function for $Nt_1 < \tau < (N + 1)t_1$ by the notation $\phi_N(x)$ with $\tau = (N + x)t_1$, $0 \leqslant x < 1$. Since an interval of length τ may span a total of $N + 1$ or $N + 2$ segments of length σ, we have

$$\phi_N(x) = \frac{(1 - p_0)^2}{1 + p_0}[(1 - x)P_{N-1} + xP_N] \tag{A.3}$$

where P_N is the probability of a message of length $N\sigma$. The recursion formula for P_N, given by Huang and Reiss, leads to $P_N = [1 + p_0(p_0)^N]/(1 + p_0)$ and the correlation function is

$$\phi_N(x) = \left(\frac{1 - p_0}{1 + p_0}\right)^2 \{1 - (-p_0)^N[1 - (1 + p_0)x]\} + \frac{1 - p_0}{1 + p_0}(1 - x)\delta_{N0} \tag{A.4}$$

The spectral density is then

$$
\begin{aligned}
G(\omega) &= 2\,\mathrm{Re}\int_0^\infty e^{-i\omega\tau}\phi(\tau)\,d\tau \\
&= 2\sigma\,\mathrm{Re}\sum_{N=0}^{\infty} e^{-i\omega\sigma N}\int_0^1 e^{-i\omega\sigma x}\phi_N(x)\,dx \\
&= 2\pi\left(\frac{1-p_0}{1+p_0}\right)^2 \delta(\omega) + \frac{1-p_0}{1+p_0}\,2\sigma\,\mathrm{Re}\int_0^1 e^{-i\omega\sigma x}(1-x)\,dx \\
&\quad - \frac{1-p_0}{1+p_0}\,2\sigma\,\mathrm{Re}\sum_{N=0}^{\infty}(-p_0e^{-i\omega\sigma})^N\int_0^1 e^{-i\omega\sigma x}[1-(1+p_0)x]\,dx \\
&= 2\pi\left(\frac{1-p_0}{1+p_0}\right)^2 \delta(\omega) + \frac{1-p_0}{1+p_0}\,\frac{4}{\omega^2\sigma} \\
&\quad \times\left[1-\cos\omega\sigma-(1-p_0)\,\mathrm{Re}\,\frac{1-e^{-i\omega\sigma}}{1+p_0e^{-i\omega\sigma}}\right] \\
&= 2\pi\left(\frac{1-p_0}{1+p_0}\right)^2 \delta(\omega) + \frac{4p_0p_1\sin^2\omega\sigma}{\omega^2\sigma(1+p_0)[1+2p_0\cos\omega\sigma+p_0^2]}
\end{aligned}
\tag{A.5}
$$

This expression is then an alternative derivation of Eq. (25) and does not agree with Eq. (27) of Huang and Reiss.[4]

For the correlated, first-order, binary code we follow Huang and Reiss[4] and assume that a 1 following a 0, or a 0 following a 1, uses a transmission time σ, while a 0 following a 0, or a 1 following a 1, uses a transmission time 2σ. We denote by P_{00}, P_{01}, P_{10}, and P_{11} the probabilities of occurrence of the corresponding digraphs. The relative probability that an arbitrary point in time lies in the interval of transmission of

a	1	following a	0	is	P_{01}
a	1	following a	1	is	$2P_{11}$
a	0	following a	1	is	$P_{01} = P_{10}$
a	0	following a	0	is	$2P_{00}$

There are three possible situations corresponding to a pulse at this arbitrary point. They are (a) the point lies in a "single length" pulse, (b) the point lies in the first half of a "double length" pulse, (c) the point lies in the second half of a "double length" pulse. The corresponding probabilities are

$$
P_a = \frac{P_{01}}{1+P_{11}+P_{00}}, \qquad P_b = P_c = \frac{P_{11}}{1+P_{11}+P_{00}} \tag{A.6}
$$

Using the same arguments as in the analysis of the uncorrelated message, we find for the correlation function in the interval $0 < \tau < \sigma$

$$\phi(\tau) = \phi_0(x) = (P_a + P_c)[(1 - x) + p_{11}x] + P_b \tag{A.7}$$

while for $\tau = (N + x)\sigma$ we can write

$$\phi(\tau) = \phi_N(x) = (P_a + P_c)[(1 - x)(P_N + P_{N+1}^{(2)}) + x(P_{N+1} + P_{N+2}^{(2)})] + P_b[(1 - x)(P_{N-1} + P_N^{(2)}) + x(P_N + P_{N+1}^{(2)})] \tag{A.8}$$

where $P_N^{(1)}$ is the frequency of a message segment of length N, terminating in a 1 which follows a 0; $P_N^{(2)}$ is the frequency of a message segment of length N, terminating in a 1 which follows a 1; $P \equiv P_N^{(1)} + P_N^{(2)}$. We define Q_N as the frequency of a message of length N, terminating in a 0.

Since all message segments we consider follow a segment that ends with a 1, we can write immediately

$$P_1 = 0; \qquad P_2^{(1)} = p_{10}p_{01}, \qquad P_2^{(2)} = p_{11}, \qquad P_2 = p_{10}p_{01} + p_{11} \tag{A.9a}$$

$$Q_1 = p_{10}; \qquad Q_2 = 0 \tag{A.9b}$$

and the recursion relations are also straightforward:

$$P_N^{(1)} = Q_{N-1}p_{01} \tag{A.10a}$$

$$P_N^{(2)} = P_{N-2}p_{11} \tag{A.10b}$$

$$Q_N = P_{N-1}p_{10} + Q_{N-2}p_{00} \tag{A.10c}$$

When we eliminate Q_N we obtain a recursion formula for P_N [in fact, since the system of Eqs. (A.10a)–(A.10c) is linear with coefficients that are independent of N, all of the quantities $P_N^{(1)}$, $P_N^{(2)}$, P_N, Q_N, and $\phi_N(x)$ satisfy the same recursion expression]:

$$P_{N+1} - (1 + p_{00}p_{11})P_{N-1} + p_{00}p_{11}P_{N-3} = 0 \tag{A.11}$$

The system is decoupled into independent solutions for N even and for N odd, both of which satisfy the same equation. When the conditions of Eq. (A.9) are imposed on the solution we find that P_N vanishes for N odd, and

$$P_{2n}^{(1)} = \frac{p_{01}p_{10}}{1 - p_{00}p_{11}}[1 - p_{00}^n p_{11}^n] \tag{A.12a}$$

$$P_{2n}^{(2)} = \frac{1}{1 - p_{00}p_{11}}[p_{01}p_{11} + p_{10}p_{00}^n p_{11}^n] \tag{A.12b}$$

When we introduce (A.12a) and (A.12b) into (A.10a)–(A.10c) and make use of relations (A.6) and (27e), we find

$$\phi_{2n}(x) = \frac{p_{01}}{2(1 - p_{00}p_{11})^2}\{p_{01}(1 + p_{11}^2) + p_{10}(p_{00} + p_{11})p_{00}^n p_{11}^n - p_{10}x[p_{10}p_{01} + (p_{11} + p_{00} - 2p_{00}p_{11})p_{00}^n p_{11}^n]\} \tag{A.13a}$$

and

$$\phi_{2n+1}(x) = \frac{p_{01}}{2(1 - p_{00}p_{11})^2}\{2p_{11}p_{01} + 2p_{10}p_{00}^{n+1}p_{11}^{n+1} + p_{10}x[p_{10}p_{01} - (2 - p_{00} - p_{11})p_{00}^{n+1}p_{11}^{n+1}]\} \tag{A.13b}$$

The two expressions can be combined to give, with $q^2 = p_{00}p_{11}$,

$$\phi_N(x) = \frac{1}{4(1 - q^2)^2}[p_{01}^2\{(1 + p_{11})^2 + (-)^N p_{10}^2(1 - 2x)\} + p_{01}p_{10}q^N\{(p_{00} + p_{11} + 2q)[1 - (1 - q)x] + (-)^N(p_{00}p_{11} - 2q)[1 - (1 + q)x]\}] \tag{A.14}$$

At long times, i.e., for $N \to \infty$, $q^N \to 0$, the persistent part of the correlation function is

$$\phi_N{}^\infty(x) = \frac{p_{01}^2}{4(1 - q^2)^2}[(1 + p_{11})^2 + (-)^N p_{10}^2(1 - 2x)] = \frac{p_{01}^2}{4(1 - q^2)^2}\left[(1 + p_{11})^2 + \frac{8p_{10}^2}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos[(2k - 1)\pi x]}{(2k - 1)^2}\right] \tag{A.15}$$

and this yields the discrete power spectrum

$$G_d(\omega) = \frac{\pi p_{01}^2}{4(1 - p_{00}p_{11})^2} \times \left[(1 + p_{11})^2\delta(\omega) + \frac{8(1 - p_{11})^2}{\pi^2}\sum_{k=1}^{\infty}\frac{\delta(\omega - [(2k - 1)/\sigma])}{(2k - 1)^2}\right] \tag{A.16}$$

The regular part of the power spectrum can be written

$$\frac{1}{\sigma}G_0(\omega) = \frac{p_{01}p_{10}}{2(1 - p_{00}p_{11})^2} \times \operatorname{Re}\sum_{N=0}^{\infty} q^N e^{-i\omega\sigma N}\int_0^1 e^{-i\omega\sigma x}[A^+(x) + (-)^N A^-(x)]\,dx = \frac{p_{01}p_{10}}{2(1 - p_{00}p_{11})^2}\operatorname{Re}\left[\int_0^1 \frac{e^{-i\omega\sigma x}A^+(x)}{1 - qe^{-i\omega\sigma}}\,dx + \int_0^1 \frac{e^{-i\omega\sigma x}A^-(x)}{1 + qe^{-i\omega\sigma}}\,dx\right] \tag{A.17}$$

where $A^{\pm}(x) = (p_{00} + p_{11} \pm 2q)[1 - (1 \mp q)x]$. Thus the evaluation of the power spectrum can be reduced to evaluating the real part of that part of either integral which is even in q. The algebra is now relatively simple and we finally obtain

$$\frac{1}{\sigma}G_0(\omega) = \frac{p_{01}p_{10}}{(1 - p_{00}p_{11})^2}\left(\frac{1 - \cos\omega\sigma}{\frac{1}{2}\omega^2\sigma^2}\right) \times \frac{(p_{00} + p_{11})(1 + p_{00}p_{11}) + 4p_{00}p_{11}\cos\omega\sigma}{(1 + p_{00}p_{11})^2 - 4p_{00}p_{11}\cos^2\omega\sigma} \tag{A.18}$$

The total power spectrum is of course $G_0(\omega) + G_d(\omega)$ from (A.18) and (A.16); this is in agreement with (32) when one sets $\xi_0 = \xi_3 = \xi^2$ and $\xi_1 = \xi_2 = \xi$, and with (35) when one sets $p_{00} = p_{11} = p_3$, $p_{01} = p_{10} = p_1$.

It should also be evident that the construction of the correlation function by a method similar to the arguments used here would not be possible for an arbitrary relation between the two times t_1 and t_0 and would have been significantly more complicated for an arbitrary pulse shape.

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